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Intersection numbers of Riemann surfaces from Gaussian matrix models

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Abstract: We consider a Gaussian random matrix theory in the presence of an external matrix source. This matrix model, after duality (a simple version of the closed/open string duality), yields a generalized Kontsevich model through an appropriate tuning of the external source. The n-point correlation functions of this theory are shown to provide the intersection numbers of the moduli space of curves with a p-spin structure, n marked points and top Chern class. This sheds some light on Witten's conjecture on the relationship with the pth-KdV equation.

Keywords: Matrix Models, Integrable Hierarchies.

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1. Introduction

The intersection theory for the moduli space of curves with marked points has been studied by various methods with the aim of proving Witten's conjectures [1] on the relationship between the structure of this space with KdV flows. In particular this has led to a solution through Kontsevich's Airy matrix model [2]. The generalizations of this theory have been also considered by Witten, who conjectured that the intersection numbers of the moduli space with top Chern class is described by the pth-KdV equation [3].

This generalization is related to the twisted N=2 supersymmetric conformal field theory, and particulary it has a close relation to the level $k = p - 2$ WZW model [4].

In previous articles [5, 6], we have found that a random matrix theory with an external source gives an alternative method for obtaining the intersection theory of the moduli space of Riemann surfaces with a p-spin structure.

We have discussed it through an N-k duality in the expectation values of the product of k characteristic polynomials of $N \times N$ matrices . After tuning appropriately the external source to a critical value, we obtain the intersection numbers of the moduli space from the Fourier transforms of the correlation functions. We had studied in the past random matrix theory with an external source as a way to tune higher edge singularities, including gap closing cases [7, 8]. Through this duality, plus tuning of the source, we show here that one obtains the generalized Kontsevich models.

In a previous article, we have computed the intersection numbers with one marked point for the generalized Kontsevich model with p-spin curves by the replica method applied to a Gaussian random matrix theory. We have also found, by an alternative method, that the Fourier transform of the one point correlation function $U(s)$ provides those intersection

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numbers and we have found an explicit agreement of the two approaches in the case of one-marked point and arbitrary genus when $p = 3$ [6]. This was checking that a Gaussian random matrix model with an external source is a dual (mirror) model to the generalized Kontsevich models of p-spin structures with one marked point. (We will also show below that it is true for $p > 3$.)

For p=2, which is Kontsevich original Airy matrix model, the intersection numbers have been computed explicitly, for a few marked points, from a Fourier transform of the correlation functions at the edge of the spectrum of the density of eigenvalues, in a random matrix model by Okounkov [9, 10] and in our previous article [5]. Here we extend the analysis to the case of $p \neq 2$, again from the edge singularities of the correlation functions in a matrix theory.

The intersection numbers are characterized by the genus g and marked points n with a "spin-structure" labelled by a non-negative integer j, $(j = 0, 1, \ldots, p-1)$. Higher Airy matrix models for curves with a p-spin structure are given by [2]

$$Z = \frac{1}{Z_0} \int dB \exp\left[\frac{1}{p+1}\mathrm{tr}(B^{p+1} - \Lambda^{p+1}) - \mathrm{tr}(B-\Lambda)\Lambda^p\right] \tag{1.1}$$

where

$$Z_0 = \int dB \exp\left[\sum_{j=0}^{p-1} \mathrm{tr}\frac{1}{2}\Lambda^j B \Lambda^{p-j-1} B\right] \tag{1.2}$$

The free energy, the logarithm of the partition function Z, is the generating function for the intersection numbers $< \prod \tau_{m,j} >$,

$$F = \sum_{d_{m,j}} < \prod_{m,j} \tau_{m,j}^{d_{m,j}} > \prod_{m,j} \frac{t_{m,j}^{d_{m,j}}}{d_{m,j}!} \tag{1.3}$$

where

$$t_{m,j} = (-p)^{\frac{j-p-m(p+2)}{2(p+1)}} \prod_{l=0}^{m-1} (lp + j + 1)\mathrm{tr}\frac{1}{\Lambda^{mp+j+1}} \tag{1.4}$$

According to Witten [3] the intersection numbers is given by the top Chern class c_T and by the first Chern class c_1 as

$$< \tau_{m_1,j_1} \cdots \tau_{m_n,j_n} >= \frac{1}{p^g} \int_{\bar{M}_{g,n}} c_T(j_1, \cdots, j_n) \prod_{k=1}^{n} c_1(\mathcal{L}_k)^{m_k} \tag{1.5}$$

with the condition which relates, for given p, the indices to the genus g of the surface,

$$(p+1)(2g-2+n) = \sum_{i=1}^{n} (pm_i + j_i + 1). \tag{1.6}$$

For more precise definitions and recent studies of the intersection numbers, we refer to the literatures [18 – 22].

The higher Airy matrix model of (1.1) is known to corresponds to a $(p-1)$ matrix model, which is conjectured to satisfy pth KdV equation. The Virasoro equations for this case are given in [23]. They have been used for computing recursively these intersection numbers; here these numbers are given directly through a generating function.

2. One point correlation function

In a previous article [5] we have used explicit integral representations for the correlation functions [12 – 14] for a Gaussian unitary ensemble (GUE) of random matrices M in the presence of an external matrix source.

The probabililty distribution for $N \times N$ Hermitian matrices is

$$P_A(M) = \frac{1}{Z_A} e^{-\frac{N}{2}\mathrm{tr}M^2 - N\mathrm{tr}MA} \tag{2.1}$$

We denote the eigenvalues of the external source matrix by a_α.

The Fourier transform of the one point correlation function, the density of state $\rho(\lambda)$, is

$$U(s) = \int \frac{d\lambda}{2\pi} e^{i\lambda s} < \mathrm{tr}\delta(\lambda - \mathrm{M}) > \tag{2.2}$$

where the average $< \cdots >$ is taken with respect to the probability distribution $P_A(M)$. An exact integral representation is known [12 – 14] for arbitrary a_j:

$$U(s) = \frac{1}{Ns} e^{\frac{N}{2}s^2} \oint \frac{du}{2\pi i} \prod_{j=1}^{N} \left(1 - \frac{s}{a_j - u}\right) e^{Nsu} \tag{2.3}$$

In the following we shall consider special cases in which the degeneracies of the distinct eigenvalues are proportional to N. In particular we consider $(p-1)$ different eigenvalues a_α, $(\alpha = 1, \ldots, p-1)$, each a_α being $\frac{N}{p-1}$ times degenerate. Then above expression reduces then to

$$U(s) = \frac{1}{Ns} e^{\frac{N}{2}s^2} \oint \frac{du}{2\pi i} \prod_{\alpha=1}^{p-1} \left(1 - \frac{s}{a_\alpha - u}\right)^{\frac{N}{p-1}} e^{Nsu}. \tag{2.4}$$

We now consider the large-N limit, in a regime in which s and u are of order $N^{-(p+1)}$. The distinct eigenvalues a_α are taken to be all of order one. Let us expand the integrand

$$U(s) = \frac{1}{Ns} e^{\frac{N}{2}s^2} \oint \frac{du}{2\pi i} \exp\left[-\frac{N}{p-1} \sum_{\alpha=1}^{p-1} \sum_{m=0}^{\infty} \left(\frac{(u+s)^m}{m a_\alpha^m} - \frac{u^m}{m a_\alpha^m}\right) + Nsu\right] \tag{2.5}$$

In the large-N limit the summation over m may be truncated at $m \leq p+1$ since higher orders vanish in that limit. We now specify the $(p-1)$ distinct eigenvalues of the external source by the $(p-1)$ conditions:

$$\sum_{\alpha}^{p-1} \frac{1}{a_\alpha^2} = p - 1, \quad \sum_{\alpha=1}^{p-1} \frac{1}{a_\alpha^m} = 0, \quad (m = 3, 4, \ldots, p)$$
$$\sum_{\alpha=1}^{p-1} \frac{1}{a_\alpha^{p+1}} \neq 0. \tag{2.6}$$

With these conditions, in the regime of interest, one has

$$U(s) = \frac{1}{Ns} \exp\left[-\frac{N}{p-1} s \left(\sum_{\alpha=1}^{p-1} \frac{1}{a_\alpha}\right)\right] \int \frac{du}{2\pi i} \exp C[u^{p+1} - (u+s)^{p+1}] \tag{2.7}$$

with

$$C = \frac{N}{p^2-1}\sum_{\alpha=1}^{p-1}\frac{1}{a_\alpha^{p+1}}. \tag{2.8}$$

The integration over u takes a more symmetric form after the shift $u \to u - \frac{1}{2}s$:

$$U(s) = \frac{1}{Ns}e^{-\frac{Ns}{p-1}\sum_{\alpha=1}^{p-1}\frac{1}{a_\alpha}}\int\frac{du}{2\pi i}e^{-Cf(u)} \tag{2.9}$$

with

$$\begin{aligned} f(u) &= \left(u+\frac{1}{2}s\right)^{p+1} - \left(u-\frac{1}{2}s\right)^{p+1} \\ &= \sum_{m=0}^{p+1}\binom{p+1}{m}\frac{s^m}{2^m}(1-(-1)^m)u^{p+1-m} \end{aligned} \tag{2.10}$$

where the interval of the integration of u is $(-i\infty, +i\infty)$.

3. Edge singularities

In the large N limit, the density of state $\rho(\lambda)$ has a finite support, and thus develops singularities at the edge of the distribution. We consider now the nature of this singularity for the p-spin case.

In the large N-limit the Green function $G(z) = \frac{1}{N}\langle \mathrm{tr}\frac{1}{\mathrm{z-M}}\rangle$ is given by a simple equation due to Pastur [16] (see also [17])

$$G(z) = \frac{1}{N}\sum_{\alpha=1}^{N}\frac{1}{z-a_\alpha-G(z)}. \tag{3.1}$$

For a source which consists of $(p-1)$ distinct eigenvalues, each of them degenerate $N/(p-1)$ times, this reads

$$G(z) = \frac{1}{p-1}\sum_{\alpha=1}^{p-1}\frac{1}{z-a_\alpha-G(z)} \tag{3.2}$$

If we expand this resolvent in powers of $\frac{1}{a_\alpha}$, and use the conditions (2.6), we find

$$\begin{aligned} G = &-\frac{1}{p-1}\left(\sum_{\alpha=1}^{p-1}\frac{1}{a_\alpha}\right) - \frac{1}{p-1}(z-G(z))\left(\sum_{\alpha=1}^{p-1}\frac{1}{a_\alpha^2}\right) \\ &-\frac{1}{p-1}\left(\sum_{\alpha=1}^{p-1}\frac{1}{a_\alpha^{p+1}}\right)(z-G)^p \end{aligned} \tag{3.3}$$

in which we have neglected terms of order $(z-G)^{p+1}$ or higher. From the first condition (2.6), we have

$$z' = -\frac{1}{p-1}\left(\sum_{\alpha=1}^{p-1}\frac{1}{a_\alpha^{p+1}}\right)(z'-G'(z')) \tag{3.4}$$

where $z' = z + \frac{1}{p-1}\sum \frac{1}{a_\alpha}$ and $G'(z') = G(z) + \frac{1}{p-1}\sum \frac{1}{a_\alpha^{p+1}}$. The Green function has a singularity proportional to $z'^{1/p}$; thus the density of state $\rho(\lambda) = -\frac{1}{\pi}\mathrm{Im}G(\lambda)$ has an edge singularity characterized by an exponent $\frac{1}{p}$.

Note that the conditions (2.6) for the a_α admit several solutions. However, all these different choices lead to the same singular behavior. In other words, the singularity exponent is independent of the location of the critical points $z_c = -\frac{1}{p-1}\sum \frac{1}{a_\alpha}$.

For instance, in the case $p = 3$, the (complex)solutions are

$$(a_1, a_2) = \pm(1, -1)$$

and

$$(a_1, a_2) = \pm\left(\frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}i, \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}i\right).$$

The dual higher Airy matrix model in (1.1) is obtained from expectation values of characteristic polynomials [5, 6], leading to

$$< \prod_{\alpha=1}^{p-1} \det(a_\alpha - iB)^{\frac{N}{p-1}} >=< \exp\left[\sum_{\alpha=1}^{p-1} \mathrm{trlog}\left(1 - \frac{iB}{a_\alpha}\right) + N\sum \log\left(\prod_{\alpha=1}^{p-1} a_\alpha\right)\right] > \quad (3.5)$$

Expanding the logarithm, with the conditions (2.6), we obtain in the large N limit, the higher Airy matrix model (1.1) as explained in [5, 6] (up to a change of the normalization factor).

4. Intersection numbers of one marked point for p-spin curves

We now derive the intersection numbers of one marked point from the asymptotic series expansion of $U(s)$.

(i) **p=2.** We begin with the simple edge of the semi-circle law. In the large N-limit, in the range in which s is of order $N^{-1/3}$, we have

$$\begin{aligned} U(s) &= \frac{1}{Ns} e^{-\frac{C}{4}s^3} \int_{-\infty}^{\infty} \frac{du}{2\pi i} e^{3Csu^2} \\ &= \frac{1}{Ns}\sqrt{\frac{\pi}{-3Cs}} e^{-\frac{C}{4}s^3} \end{aligned} \quad (4.1)$$

where $C = -\frac{N}{3}$. By the change of the normalization due to the higher Airy matrix model of (1.1), we have $\tilde{s}^3/24 = -Ns^3/12$, and we get

$$U = \frac{1}{Ns}\sqrt{\frac{\pi}{-3Cs}} \sum_{g=0}^{\infty} \frac{\tilde{s}^g}{(24)^g g!} \quad (4.2)$$

Noting that s is a conjugate variable to Λ, this yields a series expansion in inverse powers of Λ. From the definition (1.3), we obtain the intersection numbers for one marked point,

$$< \tau_{3g+1} >_g = \frac{1}{(24)^g g!}. \quad (4.3)$$

which agrees with our previous result [6] based on the replica method.

(ii) **p=3.** The critical point now corresponds to a density of states whose support consists of two disconnected segments, in the limit in which the gap closes. The intersection numbers with one marked point for p=3 have been obtained, for arbitrary genus g, in our previous replica article [6]:

$$< \tau_{\frac{8g-5-j}{3},j} >_g = \frac{1}{(12)^g g!} \frac{\Gamma(\frac{g+1}{3})}{\Gamma(\frac{2-j}{3})}, \tag{4.4}$$

where the spin-index is $j = 0$ for $g = 3m + 1$ and $j = 1$ for $g = 3m$ (m=1,2,3,...).

Near the edge point, $z_c = -\frac{1}{p-1} \sum \frac{1}{a_\alpha}$, we find in the scaling region from (2.9)

$$U(s) = \frac{1}{Ns(3Ns)^{1/3}} Ai(\zeta) \tag{4.5}$$

where $\zeta = -N^{2/3}(4 \cdot 3^{1/3})^{-1} s^{8/3}$ which was already established in [6]. From the standard asymptotic expansion of Airy functions $Ai(\zeta)$, we have two distinct series,

$$\begin{aligned} U(s) = \frac{1}{Ns(3Ns)^{1/3}} \Bigg[& Ai(0) \left(1 + \frac{1}{3!}\zeta^3 + \frac{1 \cdot 4}{6!}\zeta^6 + \frac{1 \cdot 4 \cdot 7}{9!}\zeta^9 + \cdots \right) \\ & + Ai'(0) \left(\zeta + \frac{2}{4!}\zeta^4 + \frac{2 \cdot 5}{7!}\zeta^7 + \frac{2 \cdot 5 \cdot 8}{10!}\zeta^{10} + \cdots \right) \Bigg] \end{aligned} \tag{4.6}$$

where $Ai(0) = 3^{-2/3}/\Gamma(\frac{2}{3})$ and $Ai'(0) = -3^{-1/3}/\Gamma(\frac{2}{3})$.

The first series in (4.6) gives the intersection numbers for $j = 1$. From this series , noting that $s \sim \frac{1}{\Lambda^3}$, we have

$$\begin{aligned} < \tau_{6,1} >_{g=3} &= \frac{1}{(12)^3 3!} \cdot \frac{1}{3}, \quad < \tau_{14,1} >_{g=6} = \frac{1}{(12)^6 6!} \cdot \frac{4}{9}, \\ < \tau_{22,1} >_{g=9} &= \frac{1}{(12)^9 9!} \cdot \frac{1 \cdot 4 \cdot 7}{3^3}, \ldots \end{aligned} \tag{4.7}$$

From the second series, we have

$$\begin{aligned} < \tau_{1,0} >_{g=1} &= \frac{1}{12}, & < \tau_{9,0} >_{g=4} &= \frac{1}{(12)^4 4!} \cdot \frac{2}{3}, \\ < \tau_{17,0} >_{g=7} &= \frac{1}{(12)^7 7!} \cdot \frac{2 \cdot 5}{3^2}, & < \tau_{25,0} >_{g=10} &= \frac{1}{(12)^{10} 10!} \cdot \frac{2 \cdot 5 \cdot 8}{3^3}. \end{aligned} \tag{4.8}$$

These values agree with (4.4) and with the results obtained by completely different methods [20].

(iii) **p = 4.** In this case, the critical values of the a_α, which satisfy the conditions (2.6), are obtained as the zeros of a cubic equation,

$$\begin{aligned} b_\alpha &= \frac{1}{a_\alpha} \quad (\alpha = 1, 2, 3) \\ (x - b_1)(x - b_2)(x - b_3) &= x^3 + \beta x^2 + \gamma x + \delta = 0 \end{aligned} \tag{4.9}$$

with

$$\beta^2 = 9 \pm 3\sqrt{6}, \quad \gamma = \frac{1}{2}(\beta^2 - 3), \quad \delta = \frac{3}{2\beta}(\beta^2 - 1) \tag{4.10}$$

There is one real solution, and two complex conjugate solutions for the a_α. (Although an analytic expression for the a_α , solutions of the cubic equation (4.9) exists , we give here the numerical values). There are two classes:

$$\begin{aligned}(a_1, a_2, a_3) &= (\pm 0.52523, \pm 0.41127 \pm 0.46403i, \pm 0.41127 \mp 0.46403i), \\ (a_1, a_2, a_3) &= (\pm 1.0076, \mp 0.71801 \pm 0.33908i, \mp 0.71801 \mp 0.33908i)\end{aligned} \tag{4.11}$$

In both cases, the density of state has one critical edge, at which it behaves as $\rho(\lambda) \sim \lambda^{1/4}$. Note that , contrary to the critical gap closing p=3 case, for p=4, there is just one single edge, similar to the p=2 case.

We have

$$\begin{aligned}U(s) &= \frac{1}{Ns} e^{-\frac{N}{3}s(\sum \frac{1}{a_\alpha}) - \frac{N}{4^2 \cdot 15}s^5 C} \int_{-\infty}^{\infty} \frac{dv}{2\pi} \exp\left[-C\left(\frac{s}{3}v^4 - \frac{s^3}{6}v^2\right)\right] \\ &= \frac{1}{Ns} e^{-\frac{N}{3}s(\sum \frac{1}{a_\alpha}) - \frac{N}{4^2 \cdot 15}s^5 C} \int_0^{\infty} \frac{dy}{2\pi} y^{-\frac{3}{4}} e^{-y + s^{5/2}\sqrt{\frac{NCy}{12}}}\end{aligned} \tag{4.12}$$

where $C = N \sum_{\alpha=1}^{3} \frac{1}{a_\alpha^5}$. In a series expansion in powers of s, we obtain

$$\begin{aligned}U(s) = &\left(\frac{3}{4Ns}\right)\left(\frac{NCs}{3}\right)^{-\frac{1}{4}} e^{-\frac{Ns}{3}\sum \frac{1}{a_\alpha}} \cdot e^{-\frac{NC}{4^2 \cdot 15}s^5} \\ &\times \left[\Gamma\left(\frac{1}{4}\right)\left(1 + \frac{1}{2!} \cdot \frac{s^5}{4}\left(\frac{NC}{12}\right) + \frac{s^{10}}{4!}\left(\frac{NC}{12}\right)^2 \frac{1 \cdot 5}{4^2} + \cdots\right)\right. \\ &\left. + s^{\frac{5}{2}}\left(\frac{NC}{12}\right)^{\frac{1}{2}} \Gamma\left(\frac{3}{4}\right)\left(1 + \frac{1}{3!}\frac{3s^5}{4}\left(\frac{NC}{12}\right) + \frac{s^{10}}{5!}\frac{4 \cdot 7}{4^2}\left(\frac{NC}{12}\right)^2 + \cdots\right)\right].\end{aligned} \tag{4.13}$$

For this $p = 4$ case, there is the overall factor $e^{-\frac{NC}{4^2 \cdot 15}s^5}$. It must also be expanded and combined with the series to find the relevant terms in the s-expansion. We obtain then the intersection numbers from this expression, with the scaling $\tilde{s}^5 = \frac{NC}{12}s^5$,

$$\begin{aligned}&< \tau_{1,0} >_{g=1} = \frac{1}{8}, && < \tau_{3,2} >_{g=2} = \frac{9}{8^2 \cdot 5!}, \\ &< \tau_{6,0} >_{g=3} = \frac{9}{8^3 \cdot 5!}, && < \tau_{8,2} >_{g=4} = \frac{7 \cdot 11}{8^5 \cdot 5! \cdot 10}, \ldots\end{aligned} \tag{4.14}$$

The results up to order $g = 4$ had been computed in our replica article [6] and indeed agree with these new results.

(iv) $\boldsymbol{p \geq 5}$. In the case $p = 5$, the solutions of the conditions (2.6) fall in three different classes.

(a) symmetric solution; $(a_1, a_2, a_3, a_4) = (\sigma + \rho i, \sigma - \rho i, -\sigma + \rho i, -\sigma - \rho)$ with $\sigma = \pm 0.776887, \rho = \pm 0.321797$.

(b) a_j given by $\pm$(0.624916, -1.01437, 0.53363 +0.473515 i, 0.53363-0.473515 i).

(c) a_j given by $\pm$(0.280577 + 0.5117 i, 0.280577-0.5117 i, 0.433665 + 0.158861 i,0.433665 - 0.158861 i).

These three cases all give a closing gap singularity with same same exponent, $\rho(\lambda) \sim |\lambda|^{1/5}$. We have

$$U(s) = \frac{1}{Ns} e^{-\frac{N}{4}(\sum \frac{1}{a_\alpha})} \int \frac{du}{2\pi} e^{-N(\sum \frac{1}{a_\alpha^6})[\frac{1}{4}su^5 + \frac{5}{24}s^3u^3 + \frac{1}{64}s^5u]} \tag{4.15}$$

The intersection numbers may be obtained from this expression in complete analogy with the p=3 case. The integrand for $p = 5$ presents five Stokes lines and the spin-label $j = 0, \cdots, 4$ characterizes the various domains with different asymptotic expansions. For $p > 5$ the situation is similar to those described above.

5. Several marked points

Up to now, we have only considered surfaces with one marked point. For higher intersection numbers we have to look at k-point correlation functions. The Fourier transform of the k-point function $U(s_1, \ldots, s_k)$ is also known in closed form; it is given by the integral representation [14].

$$\begin{aligned} U(s_1, \cdots, s_k) &= \frac{1}{n} \langle \mathrm{tr} e^{s_1 B} \cdots \mathrm{tr} e^{s_k B} \rangle \\ &= (-1)^{k(k-1)/2} e^{\sum_1^k \frac{s_i^2}{2}} \oint \prod_1^k \frac{du_i}{2i\pi} e^{\sum_1^k (u_i s_i)} \prod_{i=1}^k \prod_{m=1}^n \left(1 + \frac{s_i}{u_i - a_m}\right) \det \frac{1}{u_i + s_i - u_j} \end{aligned} \tag{5.1}$$

For the two-point function (k=2), with the same degenerate external source used in (2.4), one has

$$\begin{aligned} U(s_1, s_2) = e^{\frac{N}{2}(s_1^2 + s_2^2)} \oint \frac{du_1 du_2}{(2\pi i)^2} \prod_{\alpha=1}^{p-1} \left(1 - \frac{s_1}{a_\alpha - u_1}\right)^{\frac{N}{p-1}} \left(1 - \frac{s_2}{a_\alpha - u_2}\right)^{\frac{N}{p-1}} \\ \times \frac{e^{N(s_1 u_1 + s_2 u_2)}}{(u_1 - u_2 + s_1)(u_1 - u_2 - s_2)} \end{aligned} \tag{5.2}$$

At the edge singularity, we expand the integrand in powers of $\frac{1}{a_\alpha}$, with again the critical constraints (2.6),

$$\begin{aligned} &\prod_{\alpha=1}^{p-1} \left(1 - \frac{s}{a_\alpha - u}\right)^{\frac{N}{p-1}} e^{\frac{N}{2}s^2 + Nsu} \\ &\qquad \sim \exp\left[- \frac{N}{p-1} \left(\sum \frac{1}{a_\alpha} \right) s - \frac{N}{(p^2-1)} \sum_{\alpha=1}^{p-1} \left(\frac{(u+s)^{p+1} - u^{p+1}}{a_\alpha^{p+1}} \right) \right] \end{aligned} \tag{5.3}$$

By the shift $u \to u - \frac{1}{2}s$, we obtain

$$U(s_1,s_2) = \int \frac{du_1 du_2}{(2\pi i)^2} e^{-\frac{N}{p-1}(\sum \frac{1}{a_\alpha})s_1 - \frac{N}{(p^2-1)}(\sum_{\alpha=1}^{p-1} \frac{1}{a_\alpha^{p+1}})((u_1+\frac{s_1}{2})^{p+1}-(u-\frac{s_1}{2})^{p+1})}$$
$$\times e^{-\frac{N}{p-1}(\sum \frac{1}{a_\alpha})s_2 - \frac{N}{(p^2-1)}(\sum_{\alpha=1}^{p-1} \frac{1}{a_\alpha^{p+1}})((u_2+\frac{s_2}{2})^{p+1}-(u_2-\frac{s_2}{2})^{p+1})}$$
$$\times \frac{1}{(u_1-u_2+\frac{s_1+s_2}{2})(u_1-u_2-\frac{s_1+s_2}{2})}. \tag{5.4}$$

Similarly for the critical k-point correlation function, for an arbitrary value of p, we have

$$U(s_1,\ldots,s_k) = \frac{1}{(2\pi i)^k} \int \prod_{i=1}^{k} du_i e^{-\frac{NC}{p^2-1}\sum_{i=1}^{k}[(u_i+\frac{s_i}{2})^{p+1}-(u_i-\frac{s_i}{2})^{p+1}]}$$
$$\times \frac{e^{-\frac{N}{p-1}(\sum \frac{1}{a_\alpha})\sum s_i}}{\det(u_i-u_j+\frac{1}{2}(s_i+s_j))} \tag{5.5}$$

with $C = \sum_{\alpha=1}^{p-1} \frac{1}{a_\alpha^{p+1}}$. The leading connected part is obtained by the longest cycles (of length k) in the expansion of the determinant.

We focus now on the case p=3 (for p=2, this integral has been already evaluated and the intersection numbers for two marked points for arbitrary genus thereby obtained in [5], (see also [10, 22]).

For the case p=3, we consider the critical values $(a_1,a_2) = (1,-1)$ for simplicity. The singularity is located at the origin (the gap closing point). We replace the denominator by the integral,

$$\int_0^\infty dx e^{-[u_1-u_2+\frac{1}{2}(s_1+s_2)]x} = \frac{1}{u_1-u_2+\frac{1}{2}(s_1+s_2)} \tag{5.6}$$

Then we have

$$U(s_1,s_2) = -\frac{e^{\frac{N}{4}(s_1^4+s_2^4)}}{s_1+s_2} \int_0^\infty dx (e^{\frac{1}{2}(s_1+s_2)x} - e^{-\frac{1}{2}(s_1+s_2)x})$$
$$\times \int_{-\infty}^{\infty} \frac{dv_1}{2\pi} e^{iNs_1v_1^3 + i(x-\frac{N}{4}s_1^3)v_1} \int_{-\infty}^{\infty} \frac{dv_2}{2\pi} e^{iNs_2v_2^3 - i(x+\frac{N}{4}s_2^3)v_2} \tag{5.7}$$

which leads to

$$U(s_1,s_2) = \frac{1}{(\tilde{s}_1+\tilde{s}_2)(3\tilde{s}_1)^{1/3}(3\tilde{s}_2)^{1/3}} \int_0^\infty dy \sinh\left(\frac{1}{2}(\tilde{s}_1+\tilde{s}_2)y\right)$$
$$\times Ai\left(\frac{y-\frac{1}{4}\tilde{s}_1^3}{(3\tilde{s}_1)^{1/3}}\right) Ai\left(-\frac{y+\frac{1}{4}\tilde{s}_2^3}{(3\tilde{s}_2)^{1/3}}\right) \tag{5.8}$$

where we have used the scaled variables $s_i = N^{-1/4}\tilde{s}_i$ and $x = N^{1/4}y$. From now on we shall drop the tilde, but all s's should read $\tilde{s}$ instead.

Note that this expression is, from its definition, symmetric under the exchange of s_1 and s_2 although at this stage it looks asymmetrical.

If we scale y as $y \to (3\tilde{s}_1)^{1/3}y$, we obtain a convenient form for expanding in powers of s_i,

$$U(s_1,s_2) = \frac{1}{(s_1+s_2)(3s_2)^{1/3}}\int_0^\infty dy \sinh\left(\frac{1}{2}(s_1+s_2)(3s_1)^{1/3}y\right) \\ Ai\left(y - \frac{1}{4\cdot 3^{1/3}}s_1^{8/3}\right)Ai\left(-\left(\frac{s_1}{s_2}\right)^{1/3}y - \frac{1}{4\cdot 3^{1/3}}s_2^{8/3}\right) \tag{5.9}$$

Note that s_i is conjugate to Λ_i in a Fourier transform. However, we have defined the dual external source model with a term $\mathrm{Tr}(B\Lambda)$. The definition (1.1) of the higher Kontsevich-Airy models involves a different power of Λ, a power 3 for $p=3$. Thus the scaling is here given by

$$s_i \sim \frac{1}{\Lambda_i^3} \tag{5.10}$$

Expanding then for large Λ_i, i.e. small s_i, the leading term is obtained by replacing the $\sinh X$ by its first coefficient X and dropping the $s_i^{8/3}$ in the Airy functions. Note that the corrections involve $s^{8/3}$ and thus are of relative order $1/\Lambda^8$.

The leading term, which is order $1/\Lambda$, is

$$\begin{aligned} U(s_1,s_2) &= \frac{1}{2}\left(\frac{s_1}{s_2}\right)^{\frac{1}{3}}\int_0^\infty dy y Ai(y)Ai\left(-\left(\frac{s_1}{s_2}\right)y\right) \\ &= \frac{1}{2}\frac{(s_1s_2)^{\frac{1}{3}}\left(s_1^{\frac{1}{3}}+s_2^{\frac{1}{3}}\right)}{(s_1+s_2)}(-Ai'(0)Ai(0)) \end{aligned} \tag{5.11}$$

where $Ai(0) = 3^{-\frac{2}{3}}\frac{1}{\Gamma(\frac{2}{3})}$ and $Ai'(0) = -3^{-\frac{1}{3}}\frac{1}{\Gamma(\frac{1}{3})}$, and thus $-Ai'(0)Ai(0) = \frac{\sqrt{3}}{6\pi}$. The calculation involves the differential equation $Ai''(y) = yAi(y)$ followed by integrations by parts.

The next order, which provides the intersection numbers for two marked points at genus one, involves three terms. The first one is the cubic correction $X^3/6$ to the linear term of $\sinh X$; the other two involve the Taylor expansion of the $s^{8/3}$ terms in the Airy functions.

$$\begin{aligned} \Delta U^{(1)}(s_1,s_2) &= \frac{1}{16}\frac{s_1(s_1+s_2)^2}{(3s_2)^{\frac{1}{3}}}\int_0^\infty dy y^3 Ai(y)Ai\left(-\left(\frac{s_1}{s_2}\right)^{\frac{1}{3}}y\right) \\ &= \frac{1}{4\cdot 3^{1/3}}(s_1s_2)^{\frac{4}{3}}(Ai'(0))^2 + \frac{1}{8\cdot 3^{1/3}}s_1s_2^{\frac{2}{3}}(s_2-s_1)J \end{aligned} \tag{5.12}$$

where J is

$$J = \int_0^\infty dy Ai(y)Ai\left(-\left(\frac{s_1}{s_2}\right)^{\frac{1}{3}}y\right). \tag{5.13}$$

$$\begin{aligned} \Delta U^{(2)}(s_1,s_2) &= -\frac{1}{8\cdot 3^{1/3}}\left(\frac{s_1}{s_2}\right)^{\frac{1}{3}}s_1^{\frac{8}{3}}\int_0^\infty dy y Ai'(y)Ai\left(-\left(\frac{s_1}{s_2}\right)^{\frac{1}{3}}y\right) \\ &= \frac{1}{8\cdot 3^{1/3}}\left(\frac{s_1}{s_2}\right)^{\frac{2}{3}}\frac{s_1^{\frac{8}{3}}s_2}{s_1+s_2}(Ai'(0))^2 + \frac{1}{8\cdot 3^{1/3}}\frac{s_1^3s_2^{\frac{2}{3}}}{s_1+s_2}J \end{aligned} \tag{5.14}$$

$$\Delta U^{(3)}(s_1,s_2) = -\frac{1}{8\cdot 3^{1/3}}\left(\frac{s_1}{s_2}\right)^{\frac{2}{3}} s_2^{\frac{8}{3}}\left(\frac{s_2}{s_1}\right)^{\frac{1}{3}}\int_0^\infty dyyAi(y)Ai'\Big(-\left(\frac{s_1}{s_2}\right)^{\frac{1}{3}}y\Big)$$
$$= \frac{1}{8\cdot 3^{1/3}}\frac{s_1^{\frac{2}{3}}s_2^3}{s_1+s_2}\left(\frac{s_2}{s_1}\right)^{\frac{1}{3}}(Ai'(0))^2 - \frac{1}{8\cdot 3^{1/3}}\left(\frac{s_1}{s_2}\right)^{\frac{1}{3}}\frac{s_1^{\frac{2}{3}}s_2^3}{s_1+s_2}J \quad (5.15)$$

The terms which are proportional to the integral J cancel beautifully when one adds the three correction terms. The addition leads to

$$\Delta U(s_1,s_2) = \frac{1}{8\cdot 3^{1/3}}(s_1s_2)^{\frac{1}{3}}(s_1^2+s_1s_2+s_2^2)(Ai'(0))^2 \quad (5.16)$$

From the scaling to Λ, we have

$$(s_1s_2)^{\frac{1}{3}}(s_1^2+s_1s_2+s_2^2) \sim \frac{1}{\Lambda_1^7\Lambda_2} + \frac{1}{\Lambda_1^4\Lambda_2^4} + \frac{1}{\Lambda_1\Lambda_2^7}. \quad (5.17)$$

They correspond to $t_{0,0} \sim \frac{1}{\Lambda}$, $t_{2,0} \sim \frac{1}{\Lambda^7}$ and $t_{1,0} \sim \frac{1}{\Lambda^4}$, and they give the intersection numbers for two marked points.

$$< \tau_{0,0}\tau_{2,0} >_{g=1} = < \tau_{1,0}^2 > = \frac{1}{12}. \quad (5.18)$$

which agrees with the solution of the Virasoro equation for $p = 3$ [23, 21]. Note that here instead of the recursive calculation of these numbers used by previous authors, we have, with the integral representation (5.9), a generating function for arbitrary genera of intersection numbers with two marked points.

6. The $p \to -1$ limit

Up to now p, which characterizes the spin-structure, was an integer larger than one. It is interesting to consider how the theory is modified when it is continued to negative values of p and in particular when $p = -1$. Since the generalized Kontsevich model involves $\frac{1}{p+1}\mathrm{tr}B^{p+1}$, the limit $p \to -1$ gives a logarithmic potential $\mathrm{trlog}B$. This logarithmic potential corresponds to the Penner model [24, 25] and it is known to be related to the Euler character χ of the Riemann surfaces.

As we have discussed, the intersection numbers do depend upon p; for instance, in the case of one marked point, we have [6]

$$< \tau_{1,0} >_{g=1} = \frac{p-1}{24}. \quad (6.1)$$

The analytic continuation over p to negative values is thus possible and in this case, it gives simply $< \tau_{1,0} >_{g=1} = -\frac{1}{12}$ in agreement with the result of [24]: $\chi(\Gamma_1^1) = -\frac{1}{12}$.

One can indeed consider continuing to negative values many of those formulae. For instance in our previous work concerning the replica limit, $n \to 0$ for $n \times n$ matrix B, we had found

$$\mathrm{limit}_{n\to 0}\frac{1}{n} < \mathrm{tr}B^l > = \frac{\Gamma(4l+1)}{4^l\Gamma(2l+2)} \quad (6.2)$$

which is indeed finite, equal to $-1/3$, in the limit $l \to -1$.

The case of $p = -1$ is particulary interesting since Witten [4] had pointed out that in the limit $k \to -3$ in the level-k gauged WZW model, the intersection numbers, defined by (1.5), become the integral of the top Chern class alone,

$$\begin{aligned} Z_g &= (-1)^g \int_{\bar{M}_g} c_T \\ &= -\chi(M_{g,j}) \end{aligned} \tag{6.3}$$

where χ is the Euler character of the manifold.

If we compare our normalizations to Witten's correspondence with the level-k gauged WZW model, our definition of p is related to k by

$$p = k + 2 \tag{6.4}$$

and the limit $k \to -3$ indeed corresponds to the $p = -1$ limit.

7. Discussion

In this article we have shown that the intersection numbers of Riemann surfaces, the moduli space of surfaces with n marked points, endowed with a p-spin structure, are obtained from a Gaussian random matrix theory with external source, at an edge point where the asymptotic density of eigenvalues exhibits a singular behavior $\rho(\lambda) \sim \lambda^{\frac{1}{p}}$. The Fourier transforms $U(s_1, \ldots, s_k)$ of the k-point correlation functions provide the intersection numbers through the conjugacy relation $s_i \sim \frac{1}{\Lambda_i^p}$.

We have found an integral representation for $U(s_1, \ldots, s_k)$ at the critical point. Witten's conjecture is that the intersection numbers may be obtained recursively through Virasoro equations. Our formula is instead a closed expression for arbitary p and arbitrary genus; it also gives the possibility to continue in p to negative values (for instance $p = -1$). It is amusing to observe that the Fourier transform $U(s_1, \ldots, s_k)$ has an expansion as products of powers of s_i, which corresponds precisely to the expansion in $t_{n,j}$ of the free energy in the generalized Kontsevich model.

Recently, this p-th generalized Kontsevich model has been discussed as an effective theory of open strings between Liouville D-branes [26, 27]. The duality, which we have discussed with external source, corresponds to the relation between closed string (gravity) and open string (gauge theory) with cosmological constants Λ. The random matrix model with external source gives thus a theory for the case of D0 branes.

A possible extension of this work would deal with the orthogonal-symplectic Gaussian matrix models with external source, which are also mutually dual [28 – 30]. This duality is based on the so-called supersymmetric representation, introduced by Efetov [31] and developped by several authors [32, 33]. The time dependent case should also be reconsidered [14, 15] at the light of intersection numbers theory. This is all left to further work.

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